STRUCTURAL ANALYSIS OF THE CONJUGATE PROBLEM OF NONSTEADY HEAT EXCHANGE AT THE STAGNATION POINT OF A STREAM

A. G. Shashkov and B. M. Khusid

UDC 536.24.01

The problem of nonsteady heat exchange at the stagnation point of a stream is discussed. Its solution, obtained in Laplace transforms, is represented in the form of a schematic diagram establishing the connection between the observed quantities.

A stream of liquid flows from infinity onto a semibounded body placed across the flow and flows out along the wall from the critical point (Fig. 1). It is known that in the vicinity of the critical point the velocity components of the potential flow of an ideal fluid are U = bx and B = −by, where b is a constant. The temperature of the fluid is $T_{f,\infty}$ while the wall temperature is $T_{b,\infty}$. It is assumed that the properties of the fluid and body do not depend on temperature. Energy dissipation in the fluid is ignored. The surface of the body is adiabatic. This allows us to assign different temperatures to the fluid and the wall. At some time, taken as t = 0, the adiabaticity of the surface of the body is removed, and uniformly distributed heat sources having a specific power $q_s(t)$ start to operate at it. Here the hydrodynamics is steady-state as before.

The solution of the hydrodynamic problem of determining the fluid velocity profile near the stagnation point is given by the following expressions (see [1, 2], for example):

$$v = \sqrt{\nu b}\, f\left(y\sqrt{\frac{b}{\nu}}\right),\quad u = bxf'\left(y\sqrt{\frac{b}{\nu}}\right),$$

where the function f(η) satisfies the equation

$$f''' + ff'' - f'^2 + 1 = 0,\quad f|_{\eta=0} = f'|_{\eta=0} = 0,\quad f'|_{\eta\to\infty} \to 1.$$

It is known that $f''(0) \simeq 1.2326$.

A distinctive feature of the leading critical point is the fact that the boundary-layer variable η depends only on y, i.e., the thickness of the boundary layer does not vary along the surface.

The mathematical formulation of the conjugate problem of heat exchange at the stagnation point of the stream is reduced to the energy equations in the fluid and the body with the corresponding boundary conditions:

$$\frac{\partial T_f}{\partial t} + bxf'\left(y\sqrt{\frac{b}{\nu}}\right)\frac{\partial T_f}{\partial x} - \sqrt{\nu b}\, f\left(y\sqrt{\frac{b}{\nu}}\right)\frac{\partial T_f}{\partial y} = a_f\left(\frac{\partial^2 T_f}{\partial x^2} + \frac{\partial^2 T_f}{\partial y^2}\right),$$

$$\frac{\partial T_b}{\partial t} = a_b\left(\frac{\partial^2 T_b}{\partial x^2} + \frac{\partial^2 T_b}{\partial y^2}\right),$$

the boundary conditions

$$T_f|_{y\to\infty} \to T_{f,\infty},\quad T_b|_{y\to-\infty} \to T_{b,\infty},$$

$$T_f|_{t=0} = T_{f,\infty},\quad T_b|_{t\to\infty} = T_{b,\infty},$$

and the conjugation conditions

$$\lambda_f \frac{\partial T_f}{\partial y}\Big|_{y=0} + \lambda_b \frac{\partial T_b}{\partial y}\Big|_{y=0} = -Q_s(t),\quad T_f|_{y=0} = T_b|_{y=0}.$$

A. V. Lykov Institute of Heat and Mass Exchange, Academy of Sciences of the Belorussian SSR. Belorussian Polytechnic Institute, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 37, No. 4, pp. 730-738, October, 1979. Original article submitted June 8, 1978.

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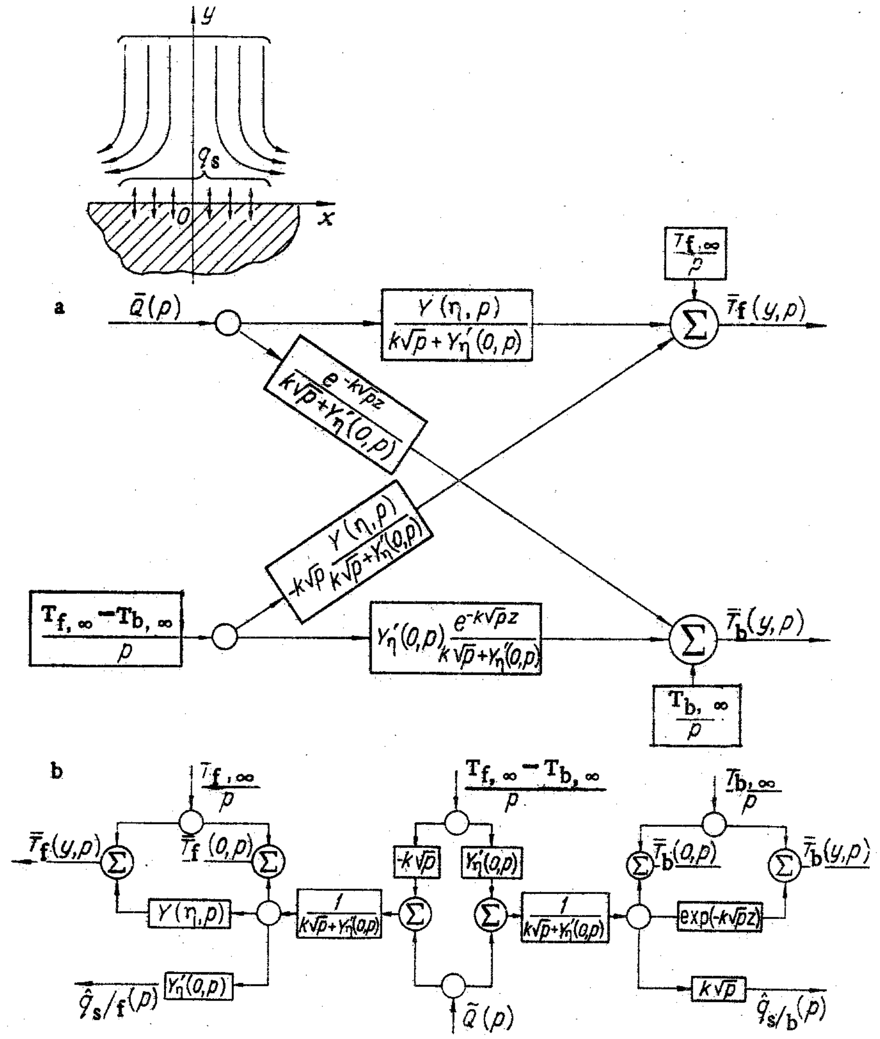

Fig. 1. Schematic diagram of conjugate problem of nonsteady heat exchange at a front critical point: a) coarse; b) fine structure; $\overline{T}_f(y, p) = \frac{T_{f,\infty}}{p} + \left[\overline{Q}(p) - \frac{T_f - T_{b,\infty}}{p} k\sqrt{p}\right] \frac{Y(\eta, p)}{k\sqrt{p} + Y'_\eta(0, p)}$;

$$\overline{T}_b(y, p) = \frac{T_{b,\infty}}{p} + \left[\overline{Q}(p) + \frac{T_{f,\infty} - T_{b,\infty}}{p} Y'_\eta(0, p)\right] \frac{e^{-k\sqrt{p}z}}{k\sqrt{p} + Y'_\eta(0, p)}.$$

As seen from these equations, the solution of the thermal problem can be sought in the form of functions $T_f = T_f(y, t)$ and $T_b = T_b(y, t)$ which do not depend on x.

Let us choose the scales for making the energy equations in the fluid and the body dimensionless. We take the scale of length in the fluid, as in the boundary layer, as $L_f = \sqrt{\nu/b}$, while in the body we take the scale L so that $\lambda_b/L_b = \lambda_f/L_f$. Hence, $L_b = (\lambda_b/\lambda_f)\sqrt{\nu/b}$. The time scale is the same for the fluid and the body, $t_0 = L_f^2/a_f = \Pr/b$; a_f and a_b are the thermal diffusivities of the fluid and the body.

In the dimensionless quantities $z = y/L_b$ (in the body), $\eta = y/L_f$ (in the fluid), and $\tau = t/t_0$, the energy equations in the fluid and the wall are written in the form (the OY axes in the body and the fluid are directed in opposite ways)

$$\frac{\partial T_f}{\partial \tau} = \frac{\partial^2 T_f}{\partial \eta^2} + \Pr f(\eta) \frac{\partial T_f}{\partial \eta}, \tag{1}$$

$$k^2 \frac{\partial T_b}{\partial \tau} = \frac{\partial^2 T_b}{\partial z^2} \tag{2}$$

with the boundary and initial conditions

$$T_f|_{\eta\to\infty} \to T_{f,\infty},\ T_f|_{\tau=0} = T_{f,\infty},\ T_b|_{z\to\infty} \to T_{b,\infty},\ T_b|_{\tau=0} = T_{b,\infty} \tag{3}$$

and the conjugation conditions at the surface of the body

$$-\left.\frac{\partial T_f}{\partial \eta}\right|_{\eta=0} - \left.\frac{\partial T_b}{\partial z}\right|_{z=0} = \frac{L_f}{\lambda_f} q_s(\tau),\ T_f|_{\eta=0} = T_b|_{z=0}. \tag{4}$$

Here $k = (\lambda_b c_b \rho_b/\lambda_f \rho_f c_f)^{1/2}$ characterizes the ratio of activities of the body and the fluid.

In view of the linearity of Eqs. (1)-(4), the solution of the thermal problem will be sought in the form

$$T_f = T_{f,\infty} + T_f^{(1)} + T_f^{(2)};\ T_b = T_{b,\infty} + T_b^{(1)} + T_b^{(2)},$$

where the fields of temperatures $T_f^{(1)}$, $T_b^{(1)}$ and $T_f^{(2)}$, $T_b^{(2)}$ are due to the differential of the initial temperatures between the fluid and the wall $(T_{f,\infty} - T_{b,\infty})$ and to the action of surface heat sources $q_s(t)$, respectively.

Applying a Laplace transform with respect to the dimensionless time

$$\hat{T}_{f,b}^{(i)} = \int_0^\infty T_{f,b} \exp(-p\tau)\, d\tau,$$

we obtain (i = 1, 2)

$$p\hat{T}_f^{(i)} = \frac{d^2\hat{T}_f^{(i)}}{d\eta^2} + \Pr f(\eta) \frac{d\hat{T}_f^{(i)}}{d\eta},\ \hat{T}_f^{(i)}|_{\eta\to\infty} \to 0, \tag{5}$$

$$k^2 p\hat{T}_b^{(i)} = \frac{d^2\hat{T}_b^{(i)}}{dz^2},\ \hat{T}_b^{(i)}|_{z\to\infty} \to 0 \tag{6}$$

with the conjugation conditions

$$\left.\frac{d\hat{T}_f^{(1)}}{d\eta}\right|_{\eta=0} + \left.\frac{d\hat{T}^{(1)}}{dz}\right|_{z=0} = 0,\ -\hat{T}_f^{(1)}|_{\eta=0} + \hat{T}_b^{(1)}|_{z=0} = \frac{T_{f,\infty} - T_{b,\infty}}{p}, \tag{7}$$

$$-\left.\frac{d\hat{T}_f^{(2)}}{d\eta}\right|_{\eta=0} - \left.\frac{d\hat{T}_b^{(2)}}{dz}\right|_{z=0} = Q(p),\ \hat{T}_f^{(2)}|_{\eta=0} = \hat{T}_b^{(2)}|_{z=0}, \tag{8}$$

where

$$Q(p) = \frac{\sqrt{\frac{\nu}{b}}}{\lambda_f} \int_0^\infty q_s(\tau) \exp(-p\tau)\, d\tau.$$

The solution of Eqs. (5)-(8) is written in the form

$$\hat{T}_f^{(1)} = -\frac{T_{f,\infty} - T_{b,\infty}}{p} \frac{k\sqrt{p}}{k\sqrt{p} + Y'_\eta(0, p)} Y(\eta, p),$$

$$\hat{T}_b^{(1)} = \frac{T_{f,\infty} - T_{b,\infty}}{p} \frac{Y'_\eta(0, p)}{k\sqrt{p} + Y'_\eta(0, p)} \exp(-k\sqrt{p}z), \tag{9}$$

$$\hat{T}_f^{(2)} = \frac{Q(p)}{k\sqrt{p} + Y'_\eta(0, p)} Y(\eta, p),$$

$$\hat{T}_b^{(2)} = \frac{Q(p)}{k\sqrt{p} + Y'_\eta(0, p)} \exp(-k\sqrt{p}z). \tag{10}$$

Here we introduce the function $Y(\eta, p)$, determined from the equation

$$\frac{d^2Y}{d\eta^2} + \Pr f(\eta) \frac{dY}{d\eta} = pY,\ Y|_{\eta=0} = 1,\ Y|_{\eta\to\infty} \to 0. \tag{11}$$

It is important to note that only one parameter Pr enters into (11).

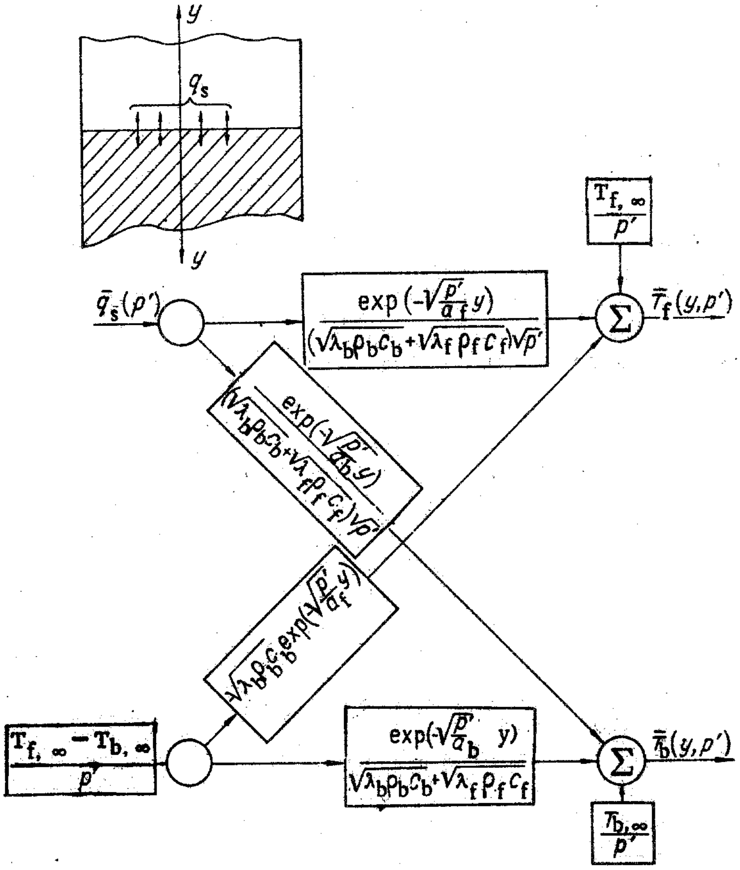

Fig. 2. Schematic diagram of the problem of conductive heat exchange of two rods with different initial temperatures and with heat release at contact surface;

$$\bar{T}_f=\frac{T_{f,\infty}}{p'}+\left[\frac{\bar{q}_s(p')}{\sqrt{p'}}-\frac{T_{f,\infty}-T_{b,\infty}}{p'}\sqrt{\lambda_b\rho_b c_b}\right]\frac{e^{-\sqrt{\frac{p'}{a_f}}y}}{\sqrt{\lambda_b\rho_b c_b}+\sqrt{\lambda_f\rho_f c_f}};$$

$$\bar{T}_b=\frac{T_{b,\infty}}{p'}+\left[\frac{\bar{q}_s(p')}{\sqrt{p'}}+\frac{T_{f,\infty}-T_{b,\infty}}{p'}\right]\frac{e^{-\sqrt{\frac{p'}{a_b}}y}}{\sqrt{\lambda_b\rho_b c_b}+\sqrt{\lambda_f\rho_f c_f}}$$

To clarify the physical meaning of $Y'_\eta(0, p)$ we calculate the coefficient of heat exchange in transforms, using Eqs. (9) and (10):

$$\hat{\alpha}_f(p)=\frac{\lambda_f\left.\frac{d\hat{T}_f}{dy}\right|_{y=0}}{\frac{T_{f,\infty}}{p}-\hat{T}_f|_{y=0}}=\frac{\lambda_f\left(\frac{d\hat{T}_f^{(1)}}{dy}+\frac{d\hat{T}_f^{(2)}}{dy}\right)\Big|_{y=0}}{-(\hat{T}_f^{(1)}+\hat{T}_f^{(2)})|_{y=0}}=\frac{\lambda_f}{L_f}Y'_\eta(0,p).$$

Thus, $Y'_\eta(0, p)$ is the dimensionless heat-exchange coefficient at the wall surface.

Equations (9) and (10), describing the temperature fields in the fluid and the body, are presented in Fig. 1 in the form of schematic diagrams which graphically show the connection between the observed quantities.

As seen from Fig. 1a, b the schematic diagram has the inputs $T_{f,\infty}/p$, $T_{b,\infty}/p$, $(T_{f,\infty}-T_{b,\infty})/p$, and $\hat{Q}(p)$. If $T_{f,\infty}=T_{b,\infty}$, e.g., then there remains only one input, $\hat{Q}(p)$, and the schematic diagram gives the temperature distribution in the field and the body due to the action of the heat source.

When $\hat{Q}(p) = 0$ we have the solution of the thermal problem describing the equalization of the temperature of the fluid and the body.

It is interesting to note that the types of schematic diagrams for the conjugate problem of convective heat exchange (Fig. 1a, b) and of conductive heat exchange of two rods (Fig. 2) are analogous. The only difference consists in the expressions for the transfer functions.

The asymptotic behavior of the function $Y(\eta, p)$ as $p \to 0$ and $p \to \infty$ ($\tau \to \infty$ and $\tau \to 0$, respectively) is investigated in detail in the Appendix.

If $p \to \infty$ ($|p| \gg [{}^1/_2 \mathrm{Pr}\, f''(0)]^{2/3}$), $Y'_\eta(0, p) \approx \sqrt{p}$ and for $\eta \ll 2|p|/\mathrm{Pr}\, f''(0)$ we have $Y(\eta, p) \approx \exp(-\sqrt{p}\eta)$.*

Thus, the process of heat exchange is initially determined only by heat conduction and does not depend on the hydrodynamic characteristics of the flow.

In dimensional form, at $t \to 0$,

$$Y(\eta, p) \approx \exp\left(-\sqrt{\frac{p'}{a_f}}\, y\right), \quad \exp(-k\sqrt{p}\, z) = \exp\left(-\sqrt{\frac{p'}{a_b}}\, y\right),$$

$$\frac{\hat{Q}(p)}{k\sqrt{p} + Y'_\eta(0, p)} = \frac{\hat{q}_s(p')}{(\sqrt{\lambda_b \rho_b c_b} + \sqrt{\lambda_f \rho_f c_f})\sqrt{p'}},$$

$$\frac{k\sqrt{p}}{k\sqrt{p} + Y'_\eta(0, p)} = \frac{(\lambda_b \rho_b c_b)^{1/2}}{(\lambda_b \rho_b c_b)^{1/2} + (\lambda_f \rho_f c_f)^{1/2}},$$

$$\frac{Y'_\eta(0, p)}{k\sqrt{p} + Y'_\eta(0, p)} = \frac{(\lambda_f \rho_f c_f)^{1/2}}{(\lambda_b \rho_b c_b)^{1/2} + (\lambda_f \rho_f c_f)^{1/2}},$$

where p' is a dimensional quantity [1/sec].

As seen from these equations, as $p \to \infty$ the schematic diagram of the conjugate problem of convective heat exchange changes into the schematic diagram of the heat exchange of two semibounded rods with different initial temperatures and with heat release at the contact surface.

If $p \to 0$ then

$$Y(\eta, p) \approx \frac{\int_\eta^\infty \exp[-\mathrm{Pr}\, F(\eta)]\, d\eta}{\int_0^\infty \exp[-\mathrm{Pr}\, F(\eta)]\, d\eta}, \quad Y'_\eta(0, p) \approx \frac{1}{\int_0^\infty \exp[-\mathrm{Pr}\, F(\eta)]\, d\eta},$$

where $F(\eta) = \int_0^\eta f(\eta)\, d\eta$.

In this case the thermal boundary layer is quasistationary, and the heat-exchange coefficient does not depend on p. As $p \to 0$ the right side of the schematic diagram (for the wall) changes into the schematic diagram for the temperature field in a semibounded rod with a boundary condition of the third kind and heat release at the outer surface.

As shown in the Appendix, at moderate Pr ($1 \leqslant \mathrm{Pr} \ll \infty$) one can use the following approximate equation for all p:

$$Y'_\eta(0, p) \approx -\left(\frac{1}{2}\mathrm{Pr}\, f''(0)\right)^{\frac{1}{3}} \frac{\mathrm{Ai}'}{\mathrm{Ai}}\left[\left(\frac{2}{\mathrm{Pr}\, f''(0)}\right)^{\frac{2}{3}} p\right].$$

APPENDIX

Let us investigate the asymptotic properties of the functions $Y(\eta, p)$ and $Y'_\eta(0, p)$ as $p \to 0$ ($\tau \to \infty$) and $p \to \infty$ ($\tau \to 0$).

1. $p \to 0$. Expanding $Y(\eta, p)$ in a series by powers of p

$$Y(\eta, p) = Y_0(\eta) + Y_1(\eta)p + \ldots$$

and substituting this series into (11), we obtain

*The $\approx$ sign denotes asymptotic equality.

$$Y_0'' + \Pr f Y_0' = 0,\ Y_0|_{\eta=0} = 1,\ Y_0|_{\eta\to\infty} \to 0,$$

$$Y_1'' + \Pr f Y_1' = Y_0,\ Y_1|_{\eta=0} = Y_1|_{\eta\to\infty} = 0.$$

Hence

$$Y_0(\eta) = \frac{\int\limits_{\eta}^{\infty} \exp[-\Pr F(\eta)]\, d\eta}{\int\limits_{0}^{\infty} \exp[-\Pr F(\eta)]\, d\eta},\ \text{where } F(\eta) = \int\limits_0^{\eta} f(\eta)\, d\eta,$$

$$Y_1(\eta) = \int\limits_0^{\infty} \left\{\int\limits_0^{\eta} Y_0(\xi)\exp[\Pr F(\xi)]\, d\xi\right\} \exp[-\Pr F(\eta)]\, d\eta - \left\{[1 - Y_0(\eta)] \int\limits_0^{\eta} Y_0(\xi)\exp[\Pr F(\xi)]\, d\xi\right\} \exp[-\Pr F(\eta)]\, d\eta.$$

From these equations we find

$$Y_0'(0) = \frac{1}{\int\limits_0^{\infty} \exp[-\Pr F(\eta)]\, d\eta},\quad Y_1'(0) = \int\limits_0^{\infty} Y_0^2(\eta)\exp[\Pr F(\eta)]\, d\eta.$$

Thus, as $p \to 0$ we have

$$Y_\eta'(0,\ p) = Y_0'(0) + Y_1'(0)\, p + \ldots, \tag{A1}$$

where $Y_0'(0) > 0$ and $Y_1'(0) > 0$. The coefficients of this series depend only on Pr. The first term of (A1) corresponds to the quasistationary boundary layer. Using a modification of the method of steepest descent for $\Pr \gg 1$, we have [3]

$$\int\limits_0^{\infty} \exp[-\Pr F(\eta)]\, d\eta \approx \sum_{n=0}^{\infty} d_n \Gamma\left(\frac{n+1}{3}\right) \Pr^{-\frac{n+1}{3}}, \tag{A2}$$

where

$$d_0 = \frac{1}{3}\left(\frac{6}{f''(0)}\right)^{1/3},\quad d_1 = \frac{1}{18 f''(0)}\left(\frac{6}{f''(0)}\right)^{2/3},\quad d_2 = \frac{1}{8\,(f''(0))^3},\ \ldots.$$

2. $p \to \infty$. The transformation

$$Y(\eta,\ p) = \Phi(\eta,\ p)\exp\left[-\frac{\Pr}{2} F(\eta)\right]$$

reduces Eq. (11) to the form

$$\Phi'' = \left(\frac{\Pr}{2} f' + \frac{\Pr^2}{4} f^2 + p\right)\Phi,\ \Phi|_{\eta=0} = 1. \tag{A3}$$

In order that $Y(\eta,\ p) \to 0$ as $\eta \to \infty$ it is required that $\Phi(\eta,\ p) \to 0$ as $\eta \to \infty$ also.

As $p \to \infty$ $(\tau \to 0)$ the temperature field is localized near the wall surface, and in place of $f'(\eta)$ and $f(\eta)$ in Eq. (A3) we can take the first terms of their expansions as $\eta \to 0$:

$$f'(\eta) = f''(0)\,\eta + O(\eta^2),\ f(\eta) = \frac{f''(0)\,\eta^2}{2} + O(\eta^3).$$

Neglecting the quantity $[\Pr f''(0)/8]\eta^4$ for moderate Pr ($\Pr \ll \infty$), from (A3) we obtain

$$\Phi'' - \left(\frac{\Pr}{2} f''(0)\,\eta + p\right)\Phi = 0,\ \Phi|_{\eta=0} = 1,\ \Phi|_{\eta\to\infty} \to 0.$$

The solution of this equation is written in the form

$$\Phi(\eta,\ p) = \frac{\mathrm{Ai}\left[\left(\frac{1}{2}\Pr f''(0)\right)^{1/3}(\eta + 2p/\Pr f''(0))\right]}{\mathrm{Ai}\left[\left(\frac{1}{2}\Pr f''(0)\right)^{1/3} 2p/\Pr f''(0)\right]},$$

where Ai is the Airy function.

Since $f(0) = 0$, we have

$$Y'_\eta(0,\ p) = -\left(\frac{1}{2}\Pr f''(0)\right)^{\frac{1}{3}} \frac{\mathrm{Ai}'}{\mathrm{Ai}}\left[\left(\frac{2}{\Pr f''(0)}\right)^{\frac{2}{3}} p\right]. \tag{A4}$$

It is interesting to note that by setting p = 0 in (A4) we obtain

$$Y'_\eta(0) \approx -\left(\frac{1}{2}\Pr f''(0)\right)^{\frac{1}{3}} \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} = \frac{3}{\Gamma\left(\frac{1}{3}\right)}\left(\frac{\Pr f''(0)}{6}\right)^{\frac{1}{3}}.$$

This result agrees well with the first term of the expansion (A2), according to which

$$Y'_0(0) = \frac{3}{(1/3)}\left(\frac{\Pr f''(0)}{6}\right)^{1/3}.$$

Thus, for moderate Pr ($1 \leq \Pr \ll \infty$) Eq. (A4) can be used for all p.

As $p \to \infty$ ($|p| \gg [{}^1/_2 \Pr f''(0)]^{2/3}$) we find $Y'_\eta(0, p) \approx \sqrt{p}$, while for $\eta \ll 2|p|/\Pr f''(0)$ we have $Y(\eta,\ p) \approx \exp(-\sqrt{p}\eta)$. These equations correspond to the case of pure heat conduction.

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HEAT PROPAGATION BY HEAT CONDUCTION IN ACTIVE LINEAR MEDIA

L. S. Eleinikova

UDC 536.2.01

A method to use the matrix A-parameter method [1] to solve linear heat-conduction problems in active media is proposed.

The system of differential equations describing the temperature and heat flux distribution in an inhomogeneous heat line (IHL) within which distributed heat and temperature sources act has the form [1]

$$\frac{\partial t}{\partial r} = -R_l q - I_l \frac{\partial q}{\partial \tau} + E, \tag{1}$$

$$\frac{\partial q}{\partial r} = -g_l t - c_l \frac{\partial t}{\partial \tau} + P. \tag{2}$$

Equations (1)-(2) form a system of so-called telegraph equations in which the effect of the internal distributed sources is taken into account. The case when the distributed temperature sources (E) and the distributed heat sources (P) are independent, i.e., are dependent on neither the temperature nor the heat flux, but at the same time can be given as functions of the coordinates or time, has been examined earlier [1]. It is shown there how a problem with given initial conditions reduces to a problem with independent heat sources. In this paper the case when the distributed sources of both E and P depend linearly on the temperature or on the heat flux (or on their time rate of change) is examined.

Let us consider the following variants:

1a) $E = R_{l+}q(r,\ \tau)$ are the distributed temperature sources proportional to the heat flux;

1b) $E = I_{l+}\partial q(r,\ \tau)/\partial\tau$ are the distributed temperature sources proportional to the time

A. V. Lykov Institute of Heat and Mass Transfer, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 37, No. 4, pp. 739-743, October, 1979. Original article submitted February 8, 1979.

0022-0841/79/3704-1242$07.50